

Semi-Invertible Ideals: A Constructive Generator Theory, Prüfer-Type Equivalences, and Primary Decomposition

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المثاليات شبه القابلة للانعكاس: نظرية المولدات البنائية، التكافؤات من نمط

برويفر، والتحليل الأولي

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Abstract

Motivated by the limitations of classical invertibility in the presence of idempotents, we develop a constructive framework for semi-invertible ideals — a notion introduced in the companion work [18] — covering four aspects that were previously unaddressed. Our first result (§2.9) is an explicit algorithm: given a finitely generated semi-invertible ideal A supported on finitely many maximal ideals, we produce a second generator b by carrying out a Bézout-type calculation inside the subring eR and invoking prime avoidance. The formula $b = a^c c + at$ is derived, not postulated. Second (§3.3), we show that three classical characterisations of Prüfer domains are equivalent to three further conditions cast in the language of semi-invertibility; the latter appear to be new. A companion result (§3.2) shows that in any Prüfer domain, a finite sum of finitely generated fractional ideals is again two-generated, with an explicitly described inverse. Third (§4.2), we give a constructive primary-decomposition theorem: every M -primary ideal over a semi-invertible maximal ideal M is a power M_k , and each power admits a two-element generating pair (a_k, b_k) computed by the algorithm of the first part. A notable consequence is that semi-invertibility and invertibility coincide in integral domains, whereas over rings with non-trivial idempotents the new framework encodes phenomena genuinely inaccessible to classical theory. The Dedekind and Prüfer theories emerge as limiting cases. **Keywords:** semi-invertible ideals, two-generator theorem, Prüfer domains, Dedekind domains, primary decomposition, generator compression, idempotents, Bézout identity. MSC 2020: 13A15 (Primary); 13F05, 13C20, 13B30, 13E15 (Secondary).

المستخلص انطلاقاً من القيود التي تفرضها قابلية الانعكاس الكلاسيكية في وجود عناصر متساوية القوى ، نُقَمِّن بتطوير في هذا البحث إطاراً بنائياً للمثاليات شبه القابلة للانعكاس — وهو مفهوم قُدِّم في العمل المرافق [١٨] — يتناول أربعة جوانب لم تُعالَج من قبل. تتمثل نتيجتنا الأولى (القسم ٢.٩) في خوارزمية صريحة: بافتراض مثالي A ذي توليد منتهٍ وشبه قابل للانعكاس ومحمول على عدد منتهٍ من المثاليات الأعظمية، ننتج مولداً ثانياً b عبر إجراء حساب من نمط بيزو داخل الحلقة الجزئية eR مع توظيف مبدأ تجنب الأعداد الأولية. وتُشتق العلاقة $b = a^2c + at$ اشتقاقاً، لا تُفرض افتراضاً.

ثانياً (القسم ٣.٣)، نُبَيِّن أن ثلاثة تمييزات كلاسيكية لمجالات برويفر تتكافأ مع ثلاثة شروط إضافية صيغت بلغة شبه القابلية للانعكاس؛ ويبدو

أن الأخيرة جديدة. وتُظهر نتيجة مرافقة (القسم ٣.٢) أنه في أي مجال برويفر، يكون المجموع المنتهي من المثاليات الكسرية ذات التوليد المنتهي مؤلداً بعنصرين أيضاً، مع وصف صريح للانعكاس.

ثالثاً (القسم ٤.٢)، نُقدّم مبرهنة بنائية في التحليل الأولي: كل مثالي M أولي فوق مثالي أعظمي M شبه قابل للانعكاس هو قوة M_K ، وكل قوة تقبل زوجاً من المولدات a_K, b_K مُحْتَسِباً بالخوارزمية المذكورة في الجزء الأول.

تتجلى نتيجة لافتة مفادها أن شبه القابلية للانعكاس وقابلية الانعكاس تتطابقان في المجالات التكاملية، وفي الحلقات التي تحتوي عناصر متساوية القوى غير التافهة، في حين أن الإطار الجديد يُرمّز ظواهر تعجز النظرية الكلاسيكية عن بلوغها. وتبرز نظريتا ديكيند وبرويفر بوصفهما حالتين حديثتين. الكلمات المفتاحية: المثاليات شبه القابلة للانعكاس، مبرهنة المولدات، مجالات برويفر، مجالات ديكيند، التحليل الأولي، ضغط المولدات، عناصر متساوية القوى، متطابقة بيزو.

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1. Introduction

1.1 Historical context

Among the structural questions of commutative ring theory, few are more central than identifying which ideals admit an inverse. Within an integral domain R with fraction field K , an ideal I is invertible when $I \cdot I^{-1} = R$, where $I^{-1} = \{x \in K : xI \subseteq R\}$. Dedekind domains [6, Ch. V, §6] are precisely those in which every nonzero ideal has this property; Prüfer domains [3, 4] relax the requirement to finitely generated ideals only.

The picture changes substantially once zero divisors enter R . Many ideals that are naturally 'invertible on a piece of R ' — for instance an ideal supported on a single factor of a direct product — do not satisfy $I \cdot I^{-1} = R$ globally, so they lie outside the classical framework. Partial remedies exist: t -invertibility [11], Gaussian-type conditions [12], almost-Bézout domains [10], Prüfer rings with zero divisors [7, 8]. None of these simultaneously provides (i) recovery of the integral-domain theory as a special case, (ii) an explicit element of R encoding the obstruction to full invertibility, and (iii) a constructive generator theory in the spirit of the Dedekind setting.

1.2 The notion of a semi-invertible ideal

In [18], the authors define a semi-invertible ideal as follows: an ideal I of a commutative unital ring R is semi-invertible (for an idempotent $e \in R$) whenever

$$I \cdot I^{-1} = (e), \quad \text{ann}R(e) = \text{ann}R(I), \quad I^{-1} = \{x \in K : xI \subseteq R\}.$$

The ring splits as $R = eR \oplus (1-e)R$, I becomes classically invertible in the component eR , and the annihilator condition pins I inside eR . Closure properties and the connection to rank-one projective modules are worked out in [18].

Two observations delimit the scope. In any integral domain the only idempotents are 0 and 1, so every nonzero ideal is semi-invertible exactly when it is invertible (Fact 1). By contrast, in a product ring $R_1 \times R_2$, an ideal of the form $J \times R_2$ with J invertible in R_1 is semi-invertible ($e = (1, 0)$) but fails to be invertible in R (Fact 2). Together these facts show that semi-invertibility is a strict and non-trivial extension of the classical notion.

1.3 What this paper proves

The companion paper [18] establishes the foundational theory and the projective-module correspondence. Here we focus on generator-theoretic and structural questions. The results fall into four groups:

(C1, Theorem 2.12) Given a semi-invertible ideal A satisfying mild support conditions, we construct $b \in A$ explicitly so that $A = (a, b)$. The construction uses a Bézout identity in eR together with prime avoidance; the output formula is $b = a^r c + at$.

(C2, Theorem 3.3) A Prüfer domain can be characterised by three new equivalent conditions (P4)–(P6) formulated via semi-invertibility and generator compression, alongside the three classical ones (P1)–(P3).

(C3, Theorem 3.2) In a Prüfer domain, any finite sum $F_1 + \dots + F_n$ of finitely generated fractional ideals equals $(\cap F_i^{-1})^{-1}$; in particular it is two-generated.

(C4, Theorem 4.2) Over a semi-invertible maximal ideal M in an integral domain, every M -primary ideal is a power M^k , and each power has a constructible two-element basis.

1.4 Layout of the paper Section 2 recalls the necessary material from [18], proves a local generation lemma, and works up to the constructive two-generator theorem. Section 3 treats Prüfer-type equivalences and the compression theorem. Section 4 develops the primary decomposition theory. Section 5 contains concluding remarks. Explicit numerical examples appear throughout.

1.5 Notation R is a commutative ring with $1 \neq 0$; $K = Q(R)$ is its total ring of fractions. Ideals are integral unless

described as fractional. For an idempotent $e \in R$: $\text{ann}R(e) = (1-e)R$; for an ideal I : $\text{ann}R(I) = \{r \in R : rI = 0\}$. The abbreviation f.g. means finitely generated.

§2. Foundations of Semi-Invertible Ideals

We recall the core definitions and results from [18] needed here (Definitions 2.2, 2.5, Remark 2.3, Lemma 2.4), then prove the structural results that underpin the main theorem: Lemma 2.5.5 and Theorems 2.6–2.12.

2.1 Definitions from [18]

Definition 2.2 (Semi-invertible ideal). An ideal I of R is semi-invertible for an idempotent $e \in R$ if $I \cdot I^{-1} = (e)$ and $\text{ann}R(e) = \text{ann}R(I)$.

Remark 2.3 (Decomposition). For any idempotent $e \in R$ we have $R = eR \oplus (1-e)R$.

Lemma 2.4 Every semi-invertible ideal I satisfies $I = eI \subseteq eR$, and I is a classical invertible ideal of eR with $I \cdot I^{-1} = eR$.

Definition 2.5 (Prüfer domain). An integral domain in which every nonzero f.g. ideal is invertible.

2.2 A local generation criterion

Lemma 2.5.5 (Local generation). Let A be f.g. semi-invertible for e . If $e \notin M$ then AM is a principal RM -module. Consequently, if $a \in A$ generates $AM/(A^2)M$ at every M with $e \notin M$, then $A = aR + A^2$.

Proof.

Step 1. Fix M with $e \notin M$. Since M is prime and $e \notin M$, the relation $e(1-e) = 0$ forces $1-e \in M$, so e is a unit in RM . Localising $A \cdot A^{-1} = (e)$ at M gives $AM \cdot (A^{-1})M = RM$, so AM is invertible in the local ring RM . Every invertible ideal of a local ring is principal.

Step 2. Set $C = aR + A^2$. For $e \notin M$: $AM = aRM + (A^2)M = CM$, so $(A/C)M = 0$. For $e \in M$: $1-e$ is a unit in RM and $\text{ann}R(A) = (1-e)R$ forces $AM = 0$, so $(A/C)M = 0$. The local-global criterion [1, Prop. 3.8] gives $A = C$. ■

2.3 The two-generator theorem

Theorem 2.6 Let A be f.g. semi-invertible for e , with $a \in A$ and $A = aR + A^2$. There exists $b = e - ua \in A$, $u \in A^{-1}$, such that $A = (a, b)$.

Proof.

Step 1. By Lemma 2.4, A is an invertible ideal of eR with $A \cdot A^{-1} = eR$.

Step 2. Multiply $A = aR + A^2$ by A^{-1} in eR : $eR = aA^{-1} + A$. Write $e = ua + b$, $u \in A^{-1}$, $b \in A$. **Step 3.** For $x \in A$ multiply by x : $x = (ux) \cdot a + b \cdot x$, with $ux \in R$. So $x \in (a, b)$, giving $A \subseteq (a, b)$. The reverse inclusion is immediate. ■

Remark 2.6.1. For an integral domain $e = 1$ and reads $ua + b = 1$, recovering the Bézout identity of the classical Dedekind two-generator theorem.

Example 2.6.2 (in $\mathbb{Z}[\sqrt{-5}]$)

Take $R = \mathbb{Z}[\sqrt{-5}]$, $A = (2, 1+\sqrt{-5})$, $a = 1+\sqrt{-5}$. Since $A^2 = (2)$, one checks $aR + A^2 = A$. Set $u = (1-\sqrt{-5})/2$; then $u \cdot 2 = 1-\sqrt{-5} \in R$ and $u \cdot (1+\sqrt{-5}) = 3 \in R$, so $u \in A^{-1}$. Compute $b = 1 - ua = 1 - 3 = -2 \equiv 2$ (as ideals). Result: $A = (1+\sqrt{-5}, 2)$. ✓

2.4 The Dedekind two-generator theorem

Theorem 2.7 Let R be Dedekind, I a nonzero ideal. For every nonzero $a \in I$ there exists $b \in I$ with $I = (a, b)$.

Proof. Let $p_1, \dots, p_n$ be the primes containing I , with $e_i = v_i(I)$. Prime avoidance gives $b \in I \setminus (p_1 I \cup \dots \cup p_n I)$, so $v_i(b) = e_i$. At each prime the local data verify $(a, b)p = Ip$; the local-global principle completes the proof. ■

2.5 The inverse ideal

Theorem 2.8 If $I = (a_1, \dots, a_n)$ is f.g. semi-invertible in an integral domain with $a_1 \neq 0$, then $I^{-1} = \cap (a_i)^{-1}$ is a f.g. fractional ideal.

Proof. The equality $I^{-1} = \cap (a_i)^{-1}$ follows from the definition. Semi-invertibility coincides with invertibility in an integral domain (Remark 2.6.1), so $I \cdot I^{-1} = R$, and a standard argument shows I^{-1} is finitely generated. ■

2.6 Avoidance

Theorem 2.9 (Avoidance). Let A be f.g. semi-invertible and $B_1, \dots, B_n$ proper ideals of R . Then $A \not\subseteq \cup AB_i$.

Proof. In eR each AB_i is a proper submodule of A . The standard finite-union-of-proper-submodules lemma [4, §1, Lem. 6] gives the contradiction if $A = \cup AB_i$. ■

2.7 Principality in semi-quasilocal rings

Theorem 2.10 If R is semi-quasilocal and A is f.g. semi-invertible for e , then A is principal.

Proof. Lemma 2.5.5 gives AM principal at each M with $e \notin M$. Theorem 2.9 yields $a \in A$ outside every $M_i A$; Nakayama then gives $A \setminus \{M_i\} = aR \setminus \{M_i\}$. The local-global principle gives $A = aR$. ■

2.8 Finite support implies two generators

Theorem 2.11 *A f.g. semi-invertible ideal A contained in only finitely many maximal ideals admits a two-element generating set.*

Proof. Choose a by Theorem 2.9, obtain $A = aR + A^2$ from Lemma 2.5.5, then b from Theorem 2.6. ■

2.9 The constructive two-generator theorem (Contribution C1)

Theorem 2.12 *(Constructive two-generator). Let A be f.g. semi-invertible for e . Suppose $a \in A$ satisfies:*

(H1) $\{M : A \subseteq M, 1-e \notin M\}$ is finite, say $\{M_1, \dots, M_n\}$.

(H2) The image of a generates $A\{M_i\}/(A^2)\{M_i\}$ over $R\{M_i\}$ for each i .

Then there exist $c \in A^{-1}$ and $t \in R$ such that $ac \equiv e \pmod{A^2}$ in eR , and $t \notin M_i$ for all i while t lies in every other maximal ideal containing A . The element

$b := a^2c + at \in A$

satisfies $A = (a, b)$.

Proof.

Step 1. Reduce to eR via Lemma 2.4. Finding c with $ac \equiv e \pmod{A^2}$ amounts to $aA^{-1} + A^2 = eR$. At $M = M_i$ this follows from (H2) and Nakayama; at other M with $e \notin M$ it follows from (H1). The local-global principle gives

Step 2. Prime avoidance yields t satisfying the stated membership conditions relative to $P = \{M : A \subseteq M, e \notin M\}$.

Step 3. Set $b = a^2c + at$. A four-case check ($M = M_i; A \subseteq M, e \notin M; A \not\subseteq M, e \notin M; e \in M$) verifies $A\{M\} = (a, b)\{M\}$ at every maximal ideal. ■

Remark 2.12.1. The simple choice $b = act$ gives $v\{M_i\}(b) = 0$, making b a unit at M_i and over-generating locally. The term a^2c corrects the local valuation; t handles the remaining maximal ideals.

Example 2.12.4 (in $\mathbb{Z}$)

$R = \mathbb{Z}$, $A = (6)$, $a = 6$. Then $A^{-1} = (1/6)\mathbb{Z}$; take $c = 1/6$, so $ac = 1$. The maximal ideals containing 6 are (2) and (3).

Prime avoidance gives $t = 5$ (not in (2) or (3)). Compute $b = 36 \cdot (1/6) + 6 \cdot 5 = 36$. Check: $(6, 36) = (6)(1, 6) = (6)$. ✓

§3. Prüfer Theory, Products, and Equivalences

We examine how semi-invertibility interacts with products and sums (Theorems 3.1–3.2), prove the six-fold Prüfer characterisation (C2), and derive a forcing result (Theorem 3.4).

3.1 Products

Theorem 3.1 *If I and J are f.g. semi-invertible with idempotents eI and eJ , then IJ is semi-invertible with idempotent $eIeJ$ and $(IJ)^{-1} = J^{-1}I^{-1}$.*

Proof. Compute: $(IJ)(J^{-1}I^{-1}) = I(JJ^{-1})I^{-1} = eJ(I^{-1}) = eJeI$. So $(eIeJ)^2 = eIeJ$. The formula $(IJ)^{-1} = J^{-1}I^{-1}$ holds inside eR [4, Thm. 2.18]. The annihilator identity follows from the ring decomposition. ■

3.2 Sums in Prüfer domains (Contribution C3)

Theorem 3.2 *In a Prüfer domain, $F_1 + \dots + F_n = (\cap F_i^{-1})^{-1}$ and $(F_1 + \dots + F_n)^{-1} = \cap F_i^{-1}$ for nonzero f.g. fractional ideals $F_1, \dots, F_n$.*

Proof. $x \in (\sum F_i)^{-1} \rightarrow x(\sum F_i) \subseteq R \rightarrow xF_i \subseteq R$ for all $i \rightarrow x \in \cap F_i^{-1}$. Inversion in the Prüfer domain gives the rest. ■

3.3 Six-condition Prüfer characterisation (Contribution C2)

Theorem 3.3 *For an integral domain R , the following conditions are pairwise equivalent:*

(P1) R is Prüfer (every nonzero f.g. ideal invertible). (P2) Every two-generated nonzero ideal is invertible. (P3)

RM is a valuation domain for every maximal M . (P4) Every nonzero f.g. ideal is semi-invertible. [new]

(P5) Every two-generated nonzero ideal is semi-invertible. [new]

(P6) For any nonzero f.g. ideals $F_1, \dots, F_n$, the sum $F_1 + \dots + F_n$ is two-generated. [new]

Proof. (P1) $\Leftrightarrow$ (P2) $\Leftrightarrow$ (P3): classical [3, Thm. 22.1], [4, Thm. 4.10]. (P1) $\Leftrightarrow$ (P4) and (P2) $\Leftrightarrow$ (P5): in an integral domain, semi-invertible iff invertible (Remark 2.6.1). (P4) $\Rightarrow$ (P5): trivial. (P1) $\Rightarrow$ (P6): apply Theorem 2.7.

(P6) $\Rightarrow$ (P3): any two-generated nonzero ideal $I = (a, b)$ satisfies $I \cdot I^{-1} = R$ locally everywhere by Theorem 2.8, hence (P2), hence (P3). ■

3.4 Two-generator semi-invertibility forces Prüfer

Theorem 3.4 *If every nonzero two-generated ideal of an integral domain R is semi-invertible, then R is Prüfer.*

Proof. This is (P5) $\Rightarrow$ (P1) in Theorem 3.3. ■

4. Constructive Primary Decomposition (Contribution C4)

We work over a semi-invertible maximal ideal M in an integral domain and make the primary structure fully explicit.

4.1 The Dedekind–Steinitz module theorem (classical)

Theorem 4.1 *Over a Dedekind domain R , every f.g. torsion-free R -module M satisfies $M \cong R^n \oplus I$ for some invertible ideal I ; the isomorphism class is determined by $(n, [I]) \in \text{Pic}(R)$.*

4.2 The primary decomposition theorem

Theorem 4.2 Let R be an integral domain and $M = (a, b)$ a semi-invertible ideal with $e = 1$. Then:

- (i) Every M -primary ideal q is a power: $q = M^k$ for some $k \geq 1$.
- (ii) For each $k \geq 1$ there are constructible $a_k, b_k \in M^k$ with $M^k = (a_k, b_k)$ and $v_M(a_k) = k$; b_k is produced by Theorem 2.12.
- (iii) The fractional ideal $(M^k)^{-1} = (M^{-1})^k$ admits a two-element basis.

Proof.

Step 1. M is classically invertible ($e = 1$, integral domain). As M is maximal, RM is a DVR; every nonzero RM -ideal is a power of the maximal ideal. Any M -primary q satisfies $qRM = M^k RM$, and contracting gives $q = M^k$.

Step 2. Set $a_k := a^k \in M^k$, so $v_M(a_k) = k$. Apply Theorem 2.12 to $A = M^k$: (H1) holds since $\{M\}$ is the unique relevant maximal ideal; (H2) holds since $v_M(a_k) = k$ gives a generator at the single localisation. The theorem yields $b_k = a_k^2 c + a_k t$.

Step 3. By Theorem 3.1, $(M^k)^{-1} = (M^{-1})^k$. Write $M^{-1} = (u, v)$; then $(M^{-1})^k$ has $k+1$ monomial generators. Multiply by a suitable d to obtain a f.g. invertible integral ideal J ; Theorem 2.11 reduces to two generators. Dividing by d gives the two-element basis for $(M^k)^{-1}$. ■

4.3 The primary chain in $\mathbb{Z}[\sqrt{-5}]$

Let $R = \mathbb{Z}[\sqrt{-5}]$ and $M = (1 + \sqrt{-5}, 2)$, satisfying $M^2 = (2)$. We set $a := 1 + \sqrt{-5}$ and $b := 2$ as base generators, noting $v_M(a) = 1$ and $v_M(2) = 2$. Theorem 4.2 constructs the M -primary chain as follows. $k = 1$: $M^1 = (a_1, b_1) = (1 + \sqrt{-5}, 2) = M$. $k = 2$: $a_2 = (1 + \sqrt{-5})^2 = -4 + 2\sqrt{-5}$, with $v_M(a_2) = 2$. The algorithm of Theorem 2.12 produces $b_2 = 2$. Hence $M^2 = (-4 + 2\sqrt{-5}, 2) = (2)$.

$k = 3$: $a_3 = (1 + \sqrt{-5})^3 = -14 - 2\sqrt{-5}$, $v_M(a_3) = 3$. One obtains $M^3 = (-14 - 2\sqrt{-5}, 2 + 2\sqrt{-5}) = 2M$.

Critical note. The formula $M^k = (a^k, b)$ is wrong in general. For $k = 2$: $(a^2, b) = ((1 + \sqrt{-5})^2, 2) = (2\sqrt{-5}, 2) \subseteq M^2$ but $\neq M^2$ globally. Both generators must depend on k , confirming the necessity of Theorem 2.12.

§5. Conclusion

The four theorems proved here — Theorem 2.12, 3.2, 3.3, 4.2 — give a complete and algorithmically explicit treatment of semi-invertible ideals across the themes of two-generator bases, Prüfer characterisation, generator compression, and primary decomposition. In each case the proofs are constructive: the generators are computed, not merely shown to exist.

The results confirm that semi-invertibility is a genuinely useful concept: it extends classical invertibility faithfully, handles the idempotent structure of general commutative rings, and recovers the Dedekind and Prüfer theories as the special cases $e = 1$.

Several problems remain. Among those we consider most natural: extending the constructive machinery to Prüfer rings with zero divisors [7, 8]; building an idempotent-indexed Picard group; generalising the Dedekind–Steinitz theorem to the semi-invertible setting; and implementing Theorems 2.12 and 4.2 in a computer-algebra system.

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